OUTER DIRICHLET PROBLEM
FOR A SEMI - INFINITE CYLINDER

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We consider an axisymmetric Dirichlet problem for the Laplace equation in a region representing a space with a semi-infinite cylindrical cut, and reduce the problem to the Fredholm integral equation of second kind. Solution of this equation is shown to exist and be unique, using the principle of a fixed point.

1. Formulation of the problem and its reduction to dual integral equations. We divide the outside of the cylinder (Fig. 1) into two regions : *1* ($z<0$, $0\leqslant r<\infty$) and *2* ($z>0$, $1<r<\infty$) and we obtain in these regions the harmonic functions $u_1(r, z)$ and $u_2(r, z)$

$$\Delta u_1 = 0, \qquad \Delta u_2 = 0 \tag{1.1}$$

satisfying the boundary conditions

$$u_1\big|_{\substack{z=0\\ r<1}} = f(r), \qquad u_2\big|_{r=1} = g(z) \tag{1.2}$$

and the conditions of continuity

$$u_1\big|_{\substack{z=0\\ r>1}} = u_2\big|_{z=0}, \qquad \frac{\partial u}{\partial z}\bigg|_{\substack{z=0\\ r>1}} = \frac{\partial u}{\partial z}\bigg|_{z=0} \tag{1.3}$$

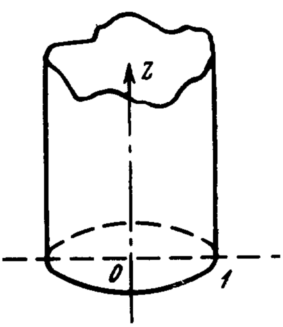

Fig. 1

Regarding the behavior at infinity we assume, that $\lim u_1 = 0$ as $z\to-\infty$, and when $z\to+\infty$, the solution u_2 should become a solution of the corresponding antisymmetric Dirichlet problem for an infinite cylinder.

We seek the harmonic functions $u_1(r, z)$ and $u_2(r, z)$ in the form of the following integrals:

$$u_1(r, z) = \int_0^\infty A(\lambda) J_0(\lambda r) e^{\lambda z} d\lambda \tag{1.4}$$

$$u_2(r, z) = \frac{2}{\pi}\int_0^\infty \frac{K_0(\nu r)}{K_0(\nu)} \sin \nu z\, d\nu \int_0^\infty g(\zeta)\sin \nu\zeta\, d\zeta + \int_0^\infty \lambda B(\lambda) \operatorname{Im}\left[\frac{H_0^{(1)}(\lambda r)}{H_0^{(1)}(\lambda)}\right] e^{-\lambda z} d\lambda \tag{1.5}$$

satisfying the indicated conditions at infinity and the second condition of (1.2). The first condition of (1.3) will become

$$\int_0^\infty A(\lambda) J_0(\lambda r)\, d\lambda = \int_0^\infty \lambda B(\lambda) \operatorname{Im}\left[\frac{H_0^{(1)}(\lambda r)}{H_0^{(1)}(\lambda)}\right] d\lambda \tag{1.6}$$

This enables us to use the Weber transforms to express $B(\lambda)$ in terms of $A(\lambda)$

$$B(\lambda) = \int_1^\infty \rho \operatorname{Im}\left[H_0^{(2)}(\lambda) H_0^{(1)}(\lambda\rho)\right] d\rho \int_0^\infty A(\nu) J_0(\nu\rho)\, d\nu \tag{1.7}$$

Fulfilling the remaining conditions of (1.2) and (1.3) we can reduce the problem to dual integrations (dual equations with the Weber kernel were considered e.g. in [1-3]) in $A(\lambda)$

$$\int_0^\infty A(\lambda) J_0(\lambda r)\, d\lambda = f(r) \qquad (0 \leqslant r < 1) \tag{1.8}$$

$$\int_0^\infty \lambda A(\lambda) J_0(\lambda r)\, d\lambda + \int_0^\infty \lambda^2 B(\lambda) \operatorname{Im}\left[\frac{H_0^{(1)}(\lambda r)}{H_0^{(1)}(\lambda)}\right] d\lambda = \frac{2}{\pi}\int_0^\infty \frac{K_0(\nu r)}{K_0(\nu)} \nu\, d\nu \int_0^\infty g(\zeta) \sin \nu\zeta\, d\zeta$$
$$(1 < r < \infty) \tag{1.9}$$

2. Reduction of the dual integral equations to the Fredholm integral equations. Let us make the following integral substitution:

$$A(\lambda) = \int_0^\infty \Phi(t) \cos \lambda t\, dt \tag{2.1}$$

We shall find that it will be more convenient to seek $\Phi(t)$ sepatately at each range $(0,\ 1)$ and $(1,\ \infty)$ of variation of t. We shall therefore use the following notation:

$$\Phi(t) = \begin{cases} \varphi(t) & (0 < t < 1) \\ \psi(t) & (1 < t < \infty) \end{cases} \tag{2.2}$$

Inserting (2. 1) into (1. 8) and using the well known relation [4]

$$\int_0^\infty J_0(\lambda r) \cos \lambda t\, d\lambda = \begin{cases} 0 & (t > r) \\ (r^2 - t^2)^{-1/2} & (t < r) \end{cases} \tag{2.3}$$

we obtain

$$\int_0^r \frac{\varphi(t)\, dt}{\sqrt{r^2 - t^2}} = f(r) \qquad (0 < r < 1) \tag{2.4}$$

This is the Schlömilch equation in $\varphi(t)$ and its solution is

$$\varphi(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{\rho f(\rho)\, d\rho}{\sqrt{t^2 - r^2}} \tag{2.5}$$

Inserting now (2. 1) into (1. 9) we obtain

$$\int_0^\infty \lambda J_0(\lambda r)\, d\lambda \int_0^\infty \Phi(\tau) \cos \lambda\tau\, d\tau = \frac{2}{\pi}\int_0^\infty \frac{K_0(\nu r)}{K_0(\nu)} \nu\, d\nu \int_0^\infty g(\zeta) \sin \nu\zeta\, d\zeta -$$
$$- \int_0^\infty \lambda^2 B(\lambda) \operatorname{Im}\left[\frac{H_0^{(1)}(\lambda r)}{H_0^{(1)}(\lambda)}\right] d\lambda \qquad (1 < r < \infty) \tag{2.6}$$

Let us apply to (2. 6) the following integral transformation: multiply each term by $2/\pi\, r(r^2 - t^2)^{-1/2}$ and integrate with respect to r from t to ∞ . Making also use of

$$\int_t^\infty \frac{J_0(\lambda r)\, r\, dr}{\sqrt{r^2 - t^2}} = \frac{\cos \lambda t}{\lambda}, \qquad \int_t^\infty \frac{K_0(\lambda r)\, r\, dr}{\sqrt{r^2 - t^2}} = \frac{\pi}{2\lambda} e^{-\lambda t}, \qquad \int_t^\infty \frac{H_0^{(1)}(\lambda r)\, r\, dr}{\sqrt{r^2 - t^2}} = \frac{e^{i\lambda t}}{\lambda} \tag{2.7}$$

we obtain (2. 6) in the form

$$\Phi(t) = \frac{2}{\pi}\int_0^\infty \frac{e^{-\nu t}}{K_0(\nu)}\, d\nu \int_0^\infty g(\zeta) \sin \nu\zeta\, d\zeta - \frac{2}{\pi}\int_0^\infty \lambda B(\lambda) \operatorname{Im}\left[\frac{e^{i\lambda t}}{H_0^{(1)}(\lambda)}\right] d\lambda \tag{2.8}$$

Let us consider (2. 8) in the region $1 < t < \infty$. Its left side is $\psi(t)$ and the first integral in its right side converges for all $g(z)$ which can be expanded into Fourier sine integral. We shall inspect the second integral in the right side of (2. 8) in more detail. To begin with

$$B(\lambda)=\int_1^\infty \rho \operatorname{Im}[H_0^{(2)}(\lambda) H_0^{(1)}(\lambda\rho)]\, d\rho \int_0^\rho \frac{\Phi(\tau)\, d\tau}{\sqrt{\rho^2-\tau^2}}=$$

$$=B_1(\lambda)+\operatorname{Im}\left[H_0^{(2)}(\lambda)\int_1^\infty \psi(\tau)\, d\tau \int_\tau^\infty \frac{H_0^{(1)}(\lambda\rho)\,\rho\, d\rho}{\sqrt{\rho^2-\tau^2}}\right] \tag{2.9}$$

where $B_1(\lambda)$ denotes the following function of λ :

$$B_1(\lambda)=\int_1^\infty \rho \operatorname{Im}[H_0^{(2)}(\lambda) H_0^{(1)}(\lambda\rho)]\, d\rho \int_0^1 \frac{\varphi(\tau)\, d\tau}{\sqrt{\rho^2-\tau^2}} \tag{2.10}$$

which is already known.

By the third integral of (2.7) we have

$$B(\lambda)=B_1(\lambda)+\frac{1}{\lambda}\int_1^\infty \psi(\tau) \operatorname{Im}[H_0^{(2)}(\lambda)\, e^{i\lambda\tau}]\, d\tau \tag{2.11}$$

hence the second integral in the right side of (2.8) can be written as

$$\int_0^\infty \lambda B(\lambda) \operatorname{Im}\frac{e^{i\lambda t}}{H_0^{(1)}(\lambda)}\, d\lambda=S(t)+\int_0^\infty \operatorname{Im}\left[\frac{e^{i\lambda t}}{H_0^{(1)}(\lambda)}\right] d\lambda \int_1^\infty \psi(\tau) \operatorname{Im}[H_0^{(2)}(\lambda)\, e^{i\lambda\tau}]\, d\tau \tag{2.12}$$

where

$$S(t)=\int_0^\infty \lambda B_1(\lambda) \operatorname{Im}\left[\frac{e^{i\lambda t}}{H_0^{(1)}(\lambda)}\right] d\lambda \tag{2.13}$$

Since

$$\operatorname{Im}\left[\frac{e^{i\lambda t}}{H_0^{(1)}(\lambda)}\right] \operatorname{Im}[H_0^{(2)}(\lambda)\, e^{i\lambda\tau}] \equiv \cos\lambda t \cos\lambda\tau-\operatorname{Re}\left[\frac{J_0(\lambda)}{H_0^{(1)}(\lambda)}\, e^{i\lambda(t+\tau)}\right] \tag{2.14}$$

expression (2.12) can be transformed into

$$\int_0^\infty \lambda B(\lambda) \operatorname{Im}\left[\frac{e^{i\lambda t}}{H_0^{(1)}(\lambda)}\right] d\lambda=S(t)+\frac{\pi}{2}\psi(t)-\operatorname{Re}\left[\int_0^\infty \frac{J_0(\lambda)}{H_0^{(1)}(\lambda)}\, e^{i\lambda t}\, d\lambda \int_1^\infty \psi(\tau)\, e^{i\lambda\tau}\, d\tau\right] \tag{2.15}$$

Let us consider the last integral on the complex variable λ-plane. Using the Cauchy's theorem we shall replace the integration along the real axis by the integration along the imaginary axis. For $\lambda = iv$ we obtain

$$\int_0^\infty \frac{J_0(\lambda)}{H_0^{(1)}(\lambda)}\, e^{i\lambda t}\, d\lambda \int_1^\infty \psi(\tau)\, e^{i\lambda\tau}\, d\tau=\frac{\pi}{2}\int_0^\infty i\,\frac{I_0(\nu)}{K_0(\nu)}\, e^{-\nu t}\, i\, d\nu \int_1^\infty \psi(\tau)\, e^{-\nu\tau}\, d\tau=$$

$$=-\frac{\pi}{2}\int_1^\infty \psi(\tau)\, d\tau \int_0^\infty \frac{I_0(\nu)}{K_0(\nu)}\, e^{-\nu(t+\tau)}\, d\nu \tag{2.16}$$

Inserting now (2.16) into (2.15) and (2.15) into (2.8), we obtain the following integral equation:

$$\psi(t)=\frac{1}{\pi}\int_0^\infty \frac{e^{-\nu t}}{K_0(\nu)}\, d\nu \int_0^\infty g(\zeta)\sin\nu\zeta\, d\zeta-\frac{1}{\pi}S(t)+\frac{1}{2}\int_1^\infty \psi(\tau)\, d\tau \int_0^\infty \frac{I_0(\nu)}{K_0(\nu)}\, e^{-\nu(t+\tau)}\, d\nu \tag{2.17}$$

Introducing a new variable

$$\omega(t)=\sqrt{t-1}\,\psi(t) \tag{2.18}$$

we obtain the following integral equation:

$$\omega(t)=h(t)-\frac{1}{2\pi}\int_1^\infty \omega(\tau)K(t,\tau)\,d\tau \tag{2.19}$$

where

$$h(t)=\frac{\sqrt{t-1}}{\pi}\left[\int_0^\infty \frac{e^{-\nu t}}{K_0(\nu)}d\nu\int_0^\infty g(\zeta)\sin\nu\zeta\,d\zeta-S(t)\right]$$

$$K(t,\tau)=\left(\frac{t-1}{\tau-1}\right)^{1/2}\int_0^\infty \Lambda(\nu)e^{-\nu(t+\tau-2)}\,d\nu,\quad \Lambda(\nu)=\pi\frac{I_0(\nu)}{K_0(\nu)}e^{-2\nu} \tag{2.20}$$

It can be shown (see the Appendix) that, if $h(t)$ is a bounded and continuous function, then Eq.(2.19) has a unique solution in this case of functions.

Thus the solution is given by Formulas (1.4)–(1.7), (2.1) and (2.2), in which the functions $\varphi(t)$ and $\psi(t)$ are defined by (2.15) and the integral equation (2.19).

3. Appendix. Let us consider the operator $y=U(\omega)$ defined by

$$y(t)=h(t)-\frac{1}{2\pi}\int_1^\infty\left[\left(\frac{t-1}{\tau-1}\right)^{1/2}\int_0^\infty\Lambda(\nu)e^{-\nu(t+\tau-2)}\,d\nu\right]\omega(\tau)\,d\tau$$

in a space of bounded, continuous functions with the norm

$$\rho(\omega,\omega^*)=\max|\omega(t)-\omega^*(t)|$$

and let us consider the modulus of the difference between y and y^* where $y^*=U(\omega^*)$

$$|y(t)-y^*(t)|\leqslant\frac{1}{2\pi}\int_1^\infty\left(\frac{t-1}{\tau-1}\right)^{1/2}\left|\int_0^\infty\Lambda(\nu)e^{-\nu(t+\tau-2)}\,d\nu\right|\cdot|\omega(\tau)-\omega^*(\tau)|\,d\tau\leqslant$$

$$\leqslant\frac{\Lambda_{max}}{2\pi}\rho(\omega,\omega^*)\int_1^\infty\frac{\sqrt{t-1}\,d\tau}{\sqrt{\tau-1}\,(t+\tau-2)}$$

The last integral is easily calculated and is equal to π, hence

$$\rho(y,y^*)\leqslant{}^1/_2\,\Lambda_{max}\,\rho(\omega,\omega^*)$$

By (2.20) $\Lambda_{max}=1.3305<2$, therefore

$$\rho(y,y^*)\geqslant\alpha\rho(\omega,\omega^*)\qquad(\alpha<1)$$

Thus the operator $y=U(\omega)$ is a contraction operator and by the Banach theorem it has a fixed point. This implies that (2.19) has a unique solution in the given class of functions, which can be obtained by the method of consecutive approximations.

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